

# VARIATION OF MIXED HODGE STRUCTURES ASSOCIATED TO AN EQUISINGULAR ONE-DIMENSIONAL FAMILY OF CALABI-YAU 3-FOLDS

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ABSTRACT. We study the variations of Mixed Hodge Structures associated to a particular one-dimensional equisingular family of Calabi-Yau threefolds arising from a one-dimensional family of quintic hypersurfaces in  $\mathbb{P}^4$  with exactly 100 ordinary double points as singularities.

## INTRODUCTION

In the 1930's, W.V.D. Hodge proved his celebrated theorem stating that for every compact Kähler manifold  $X$ , its complex de Rham cohomology  $H^k(X, \mathbb{C})$  splits as a direct sum of spaces  $H^{p,q}(\cong H^q(X, \Omega_X^p))$ , where  $p+q = k$ , called nowadays the Hodge decomposition of  $H^k(X, \mathbb{C})$  (see [9]). Remember that  $H^k(X, \mathbb{C}) \cong H^k(X, \mathbb{Z}) \otimes \mathbb{C}$ . The pair  $(H^k(X, \mathbb{Z}), \{H^{p,q}\})$  is called a (pure) Hodge structure of weight  $k$ . All varieties will be considered algebraic and defined over the complex numbers  $\mathbb{C}$ .

Another way of looking at a Hodge structure is to consider the associated Hodge filtration

$$F^j H^k(X, \mathbb{C}) := \bigoplus_{p \geq j} H^{p,q}$$

and the pair  $(H^k(X, \mathbb{Z}), \{F^j H^k(X, \mathbb{C})\})$ .

If  $X$  is projective, smooth of dimension  $n$ , then the only interesting cohomology group is  $H^n(X, \mathbb{C})$  and because of Lefschetz' theorem, we only need to consider the so called primitive cohomology  $PH^n(X, \mathbb{C}) = \{\eta \in H^n(X, \mathbb{C}) \mid \eta \cdot H = 0\}$ , where  $H$  is the class of a hyperplane section on the corresponding projective space.

In the particular case of a smooth projective hypersurface, Griffiths studied the (pure) Hodge structure of  $X$  and gave a description of it in terms of its Jacobian ring (see [6]). More precisely, let  $X = V(f) \subset \mathbb{P}^{n+1}$  with  $f$  an homogeneous polynomial of degree  $d$ . Then the space

$$(1) \quad V_k \stackrel{\text{def}}{=} \left\{ \left[ \frac{P\Omega}{f^k} \right] \in A_k^{n+1} \bmod dA_{k-1}^n \mid \deg(P) = kd - (n+2) \right\},$$

of closed  $(n+1)$ -forms with a pole of order  $k$  along  $X$ , modulo the forms  $d\eta$ , where  $\eta$  is a rational  $n$ -form with a pole of order  $k-1$  along  $X$ , can be identified via the

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residue map with the space  $F^{n-k}PH^n(X, \mathbb{C})$ . Here  $PH$  denotes the primitive cohomology and  $\Omega = \sum_{i=0}^{n+1} (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_{n+1}$  which is the fundamental homogeneous differential form obtained by contracting the volume form on  $\mathbb{C}^{n+2}$  with the Euler vector field  $\sum_{i=0}^{n+1} x_i \frac{\partial}{\partial x_i}$ .

The natural inclusion  $F^{n-k}PH^n(X, \mathbb{C}) \subset F^{n-k+1}PH^n(X, \mathbb{C})$  corresponds to the natural inclusion  $V_k \longrightarrow V_{k+1}$  given by  $\frac{P\Omega}{f^k} \mapsto \frac{fP\Omega}{f^{k+1}}$ .

Moreover, if  $J(f)$  is the Jacobian ideal of  $f$ , generated by the partial derivatives of  $f$  and  $R_f := \mathbb{C}[X_0, \dots, X_n]/J(f)$  is the Jacobian ring of  $f$ , then the above identification induces isomorphisms between  $(R_f)_{(k+1)d-n-2}$  the graded submodule of  $R_f$  of degree  $(k+1)d-n-2$  and  $PH^{n-k,k}(X, \mathbb{C})$ . In particular, the dimension of  $H^n(X, \mathbb{C})$  is independent of  $X$  itself and only depends on  $n$  and on the degree of  $f$ .

If we now consider a smooth family  $\pi : \mathcal{X} \longrightarrow B \subset \mathbb{P}^1$ , over an open set  $B$ , then on every fiber  $X_t$  one has a Hodge structure  $F^p H^k(X_t, \mathbb{C})$  and these extend to a global Hodge filtration  $\mathcal{F}^p \mathcal{H}^k$ , where  $\mathcal{H}^k \stackrel{\text{def}}{=} R^k \pi_* \mathbb{C} \otimes \mathcal{O}_B$ . It is well known that the monodromy of the family gives rise to a connection, called the Gauss-Manin connection

$$\nabla : \mathcal{H}^k \longrightarrow \mathcal{H}^k \otimes \Omega_B$$

which is compatible with the Hodge filtration. More explicitly, the Gauss-Manin connection satisfies the Griffiths transversality condition

$$\nabla : \mathcal{F}^p \mathcal{H}^k \longrightarrow \mathcal{F}^{p-1} \mathcal{H}^k \otimes \Omega_B.$$

Recall that the associated *Higgs Bundle* is given by  $\mathcal{E} = \bigoplus_{q=0}^n \mathcal{E}^{n-q,q}$ , where

$$\mathcal{E}^{n-q,q} \stackrel{\text{def}}{=} \mathcal{F}^q / \mathcal{F}^{q+1},$$

induces by means of the Gauss-Manin connection mappings:

$$\nabla : \mathcal{E}^{n-q,q} \rightarrow \mathcal{E}^{n-q+1,q-1} \otimes \Omega_B^1.$$

and fibrewise homomorphisms

$$\mathcal{E}_t^{n-q,q} \longrightarrow \mathcal{E}_t^{n-q+1,q-1}.$$

For singular varieties, Deligne developed at the early 1970's the theory of mixed Hodge structures (see [5]), which involves in general the existence of a good desingularization due to Hironaka.

Griffiths and others have tried to give an alternative description for the mixed Hodge structure of a singular variety in some cases. The most important case for us is that of a singular projective hypersurface on the projective space with isolated singularities, the simplest of which is only nodes as singularities. Griffiths [7] and later on Steenbrik [12] gave a description of the relevant cohomology group of its

proper transform under normalization in terms of the Jacobian ring of the polynomial defining it. More precisely, let  $X = V(f) \subset \mathbb{P}^{n+1}$  be a hypersurface, with  $f$  an homogeneous polynomial of degree  $d$ ;  $\Sigma \stackrel{\text{def}}{=} \text{Sing}(X)$  its singular locus and assume  $\Sigma$  consist only of  $m$  nodes and let  $\tilde{X}$  be its proper transform under normalization. Then, if  $Y \subset \mathbb{P}^{n+1}$  is a smooth hypersurface of the same degree as  $X$  one has

$$\begin{aligned} \text{rank } H_n(X) &= \text{rank } H_n(Y) - m \quad \text{and} \\ \text{rank } H_n(\tilde{X}) &= \text{rank } H_n(Y) - 2m. \end{aligned}$$

Since in this case  $\tilde{X}$  is smooth, Poincaré duality implies that

$$(2) \quad \dim H^n(\tilde{X}, \mathbb{C}) = \dim H^n(Y, \mathbb{C}) - 2m.$$

**Remark 1.** *The vector space*

$$(3) \quad V_1 = \left\{ \left[ \frac{P\Omega}{f} \right] \in A_1^{n+1} \mid \deg(P) = d - (n+2) \right\},$$

can be identified with  $F^n PH^n(\tilde{X}, \mathbb{C})$  via the residue map, whereas the space

$$V_2|_{\Sigma} = \left\{ \left[ \frac{P\Omega}{f^2} \right] \in A_2^{n+1} \bmod dA_1^n \mid \deg(P) = 2d - (n+2) \text{ and } P(Q) = 0 \forall Q \in \Sigma \right\},$$

given by the first adjunction condition on  $A_2^{n+1}$  can be identified with  $F^{n-1} PH^n(\tilde{X}, \mathbb{C})$  and the inclusion  $F^n PH^n(\tilde{X}, \mathbb{C}) \subset F^{n-1} PH^n(\tilde{X}, \mathbb{C})$  corresponds to the natural map

$$\begin{aligned} V_1 &\longrightarrow V_2|_{\Sigma} \\ \text{given by } \frac{P\Omega}{f} &\mapsto \frac{fP\Omega}{f^2}. \end{aligned}$$

Remark 1 is not particular to poles of order one and two as well as the first adjoint condition. We want to generalize to understand this remark as follows: Once again we also follow standard notation (see also [6] part II), let  $\Omega_{\mathbb{P}^4}^4(X_t)$  be the sheaf on  $\mathbb{P}^4$  of four-forms with a pole of order  $k$  along the hypersurface  $X_t$  or shortly  $\Omega_{\mathbb{P}^4}^4(k)$  and for a polynomial  $F$  we denote by  $\mu_p(F)$  the multiplicity of  $F$  in  $P$  (see [8]). Let also  $\Omega_{\mathbb{P}^4}^4(nX_t, m\Sigma_t)$  be the subsheaf of  $\Omega_{\mathbb{P}^4}^4(n)$  of four-rational forms with a pole of order  $n$  on  $X_t$  and have multiplicity greater than  $m$ . It follows that  $H^0(\mathbb{P}^4, \Omega_{\mathbb{P}^4}^4(k)) = A_k^4(X_t)$ .

In what follows we will assume that for the singular locus of  $X_t$  all singularities are of the same type.

**Definition 2.** *Given  $f \in \mathbb{C}[y_0, \dots, y_n]$  the  $m$ -adjoint condition on  $f$  relative to a subset  $T \subset X_t \subset \mathbb{P}^n$  is given by  $m = \mu_p(f)$  for all  $p \in T$ . Note that if  $m = 1$  we have only one condition, namely  $f|_T = 0$  which is equivalent to  $T \subset V(f)$ .*

**Example 3.** *The simplest example is obviously that given by the singlet  $P$  defined by a point, in that case the first adjoint condition relative to  $P$  is simply that  $f$  vanishes on  $P$ .*

**Definition 4.** *The  $n$ -adjoint space of four-rational forms with poles of order  $m$  is defined as follows:  $A_m^4(X_t, n\Sigma_t) = \{\psi \in A_m^4(X_t) \mid \psi = \frac{h\Omega}{f_t^m}, h \text{ is } n\text{-adjoint relative to } \Sigma_t\}$ .*

It follows that  $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^4(nX_t, m\Sigma_t)) = A_n^4(X_t, m\Sigma_t)$ . Clearly  $n \leq d = \deg(f)$ .

**Remark 5.** *In this sense given  $G$  a finite subset of polynomials one can generalize the adjointness condition relative to  $T$  if for all  $h \in G$  the  $m$ -adjoint condition is satisfied on  $h$ .*

In fact:

**Remark 6.** *If  $T = \Sigma_t$  the set  $\{\frac{\partial f}{\partial y_0}, \dots, \frac{\partial f}{\partial y_n}\}$  is one-adjoint relative to  $\Sigma_t$  iff  $\Sigma_t$  consists of ordinary double points.*

For  $m = 2$  if  $\Sigma_t$  consists of ordinary double points then  $dA_1^3(X_t) \subset A_2^4(X_t, \Sigma_t)$  but in general it need not be true that  $dA_{m-1}^3(X_t) \subset A_m^4(X_t, (m-1)\Sigma_t)$ . If we introduce the shorter notation  $A_m^n$  for  $A_m^4(X_t, n\Sigma_t)$  then we can define the following quotient:  $\mathcal{K}_m^n = A_m^n / dA_{m-1}^3 \cap A_m^n$ .

Let us return to the sheaf theoretic version of forms with pole order and adjointness conditions:

**Definition 7.**  $\Omega_{\mathbb{P}}^4(n, m)$  is the subsheaf on  $\mathbb{P}^4$  of four-rational forms with a pole of order  $n$  on  $X_t$  and are at least  $m$ -adjoint relative to  $\Sigma_t$

**Proposition 8.** *For  $N = 2n - 3$  positive and  $m \geq N$  then  $\pi^*(\Omega_{\tilde{X}_t}(n, m)) \subset \Omega_{\tilde{X}_t}(n)$  for  $m \geq N$  where  $\pi : \tilde{X}_t \rightarrow X_t$  is the blow-up of  $X_t$  along the center  $\Sigma_t$ .*

PROOF. This is a local computation; for that we introduce the following notation. We define a local chart in  $\mathbb{A}^4$  with coordinates  $z_1, z_2, z_3, z_4$  in a polydisc  $D_\varepsilon$  and fix a point  $P \in D_\varepsilon \cap \Sigma_t$ . It is easy to see that wlog locally around this  $D_\varepsilon$ ,  $X_t = \{z | z \cdot z = 0\}$ . We can also wlog assume that in this affine chart  $P = (0, 0, 0, 0)$ . Moreover, let  $\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4)$ . Note that  $D_\varepsilon \subset \mathbb{P}_3$  where  $\mathbb{P}_3 = \mathbb{P}(\mathbb{A}^4)$  gives rise to the strict transform:  $\tilde{D}_\varepsilon = \text{Bl}_P(D_\varepsilon) = \pi^{-1}(D_\varepsilon)$ . In order to give a local description for  $\pi$  we blow up the polydisc and obtain  $\tilde{D}_\varepsilon = \{(z, \xi) \in D_\varepsilon \times \mathbb{P}_3 | z_i \xi_j - z_j \xi_i = 0 \text{ for } 0 \leq i < j \leq 4\}$ . Also wlog  $\xi_4 \neq 0$  and in fact  $z_i = z_4 \frac{\xi_i}{\xi_4}$  for  $i = 1, 2, 3, 4$ . We further define  $u = z_4$  and  $v_i = \frac{\xi_i}{\xi_4}$  hence:

$$z_1 = v_1 u, z_2 = v_2 u, z_3 = v_3 u, z_4 = u.$$

Obviously  $\pi : \tilde{D}_\varepsilon \rightarrow D_\varepsilon$  is the projection in the first coordinate, namely  $(z, \xi) \mapsto z$  but with our local description:  $\pi(uv, v) = uv$  where  $v = (v_1, v_2, v_3, 1)$  (Note that [6] on page 523 writes incorrectly  $\pi(u, v)$ ).

Let  $\varphi \in A_n^4(X_t, m\Sigma_t)$ , in particular  $\varphi(z) = \frac{F(z)\Omega}{(z \cdot z)^n}$  such that  $F$  is  $m$ -adjoint relative to  $\Sigma_t$ . Note that  $(z \cdot z)^n = u^{2n}(1 + v \cdot v)^n$  and  $\pi^*(\varphi)(u, v) = \varphi(\pi(uv, v)) = \frac{u^3 F(uv_1, uv_2, uv_3, u) dv du}{u^{2n}(1+v \cdot v)^n}$ . The condition that there exists poles of order  $n$  is that  $\mu_p(F) \geq N$  and that  $\pi^*\varphi$  has no poles along the exceptional divisor  $\tilde{H} = \{u = 0\}$ . It is illustrative to compute just a few values for  $n, N$  given as follows:

n	2	3	4	5	6	7	8
N	1	3	5	7	9	11	13

and obtain Griffiths remark on p. 522 of *loc.cit* for  $n = 2$  as a particular case of our  $N$ . Q.E.D.

**Lemma 9.** *Let  $X = V(f) \subset \mathbb{P}^4$  and  $\Sigma$  be as before. Then  $m \leq h^{2,1}(Y)$ , where  $Y$  is any smooth hypersurface of degree  $d$  on  $\mathbb{P}^4$ .*

PROOF. Let  $X = V(f) \subset \mathbb{P}^4$  be given by an homogeneous polynomial  $f$  of degree  $d$  and  $\Sigma$  consist of precisely  $m$  nodes, then the relevant cohomology group of  $\tilde{X}$  is  $H^3$  and we have just given a nice description of  $F^3 H^3(\tilde{X}, \mathbb{C})$  and  $F^2 H^3(\tilde{X}, \mathbb{C})$ . Since  $F^0 H^3 = H^3$  we are almost done. In fact we are already done, observing that:

$$\begin{aligned} H^{0,3} &\cong E^{3,0} = F^0/F^1, \\ H^{1,2} &\cong E^{2,1} = F^1/F^2, \\ H^{2,1} &\cong E^{1,2} = F^2/F^3 \quad \text{and} \\ H^{3,0} &\cong E^{0,3} = F^3. \end{aligned}$$

Since  $\tilde{X}$  is smooth, Hodge's theorem tell us that  $\overline{H^{0,3}} = H^{3,0}$  and  $\overline{H^{1,2}} = H^{2,1}$ . According to (1 and 3),  $F^3 PH^3(\tilde{X}, \mathbb{C})$  is isomorphic to  $F^3 PH^3(Y, \mathbb{C})$ , where  $Y$  is any smooth hypersurface on  $\mathbb{P}^4$  of degree  $d$ . In particular  $H^{0,3}(\tilde{X}) \cong H^{0,3}(Y)$  and therefore, by conjugation,  $H^{3,0}(\tilde{X}) \cong H^{3,0}(Y)$ . Since we have seen (2) that  $h^3(\tilde{X}) = h^3(Y) - 2m$  and  $h^3 = h^{0,3} + h^{1,2} + h^{2,1} + h^{3,0}$ , then the difference should come from  $h^{2,1}$  and  $h^{1,2}$  which, since the corresponding spaces are dual to one another, are equal, i.e.,  $h^{2,1}(\tilde{X}) = h^{2,1}(Y) - m$  and  $h^{1,2}(\tilde{X}) = h^{1,2}(Y) - m$ . In particular, the number of nodes cannot exceed the hodge number  $h^{2,1}(Y)$  for a smooth hypersurface  $Y$  of the same degree as  $X$ . Q.E.D.

**Remark 10.** *In particular for a quintic hypersurface on  $\mathbb{P}^4$  we obtain the nice bound  $m \leq 101$ . In this case (see [1], [10] and [13]) this bound is almost sharp.*

A similar result can be proved for surfaces on  $\mathbb{P}^3$  and curves on  $\mathbb{P}^2$ .

Can we also describe the MHS of the cohomology  $H^3(X, \mathbb{C})$ ?

In the sequel, we will consider a pencil of hypersurfaces of degree  $d$  on  $\mathbb{P}^4$ , parametrized by  $\mathbb{P}^1$ , which is equisingular over some open subset  $B \subset \mathbb{P}^1$ , with precisely  $m$  nodes as the singular locus of each fiber, and will give a description of the variation of MHS associated to this family (see [5]).

## 1. GENERALIZED HODGE NUMBERS

Following Danilov and Khovanskii (see [3] §1, in particular definition 1.5 proposition 1.8, corollary 1.9 and 1.10), we define the generalized Hodge numbers:

$$e^{p,q} = e^{p,q}(X) \stackrel{\text{def}}{=} \sum_k (-1)^k h^{p,q}(H_c^k(X))$$

as well as the generalized Euler characteristic polynomial

$$e(X; x, \bar{x}) \stackrel{\text{def}}{=} \sum_{p,q} e^{p,q}(X) x^p \bar{x}^q$$

which in the sequel we will simply write  $e(X)$ . We summarize some well known results about this polynomial (see [3]) in a single lemma.

**Lemma 11.**

- Suppose  $X$  is a disjoint union of a finite number of locally closed subvarieties  $X_i$ ,  $i \in I$ . Then  $e(X) = \sum_i e(X_i)$ .
- If  $f : X \longrightarrow Y$  is a bundle with fiber  $F$  which is locally trivial in the Zariski topology, then  $e(X) = e(Y) \times e(F)$ .
- If  $X$  is a point, then  $e(X) = 1$ .
- $e(\mathbb{P}^1) = 1 + x\bar{x}$ .
- $e(\mathbb{P}^n) = 1 + x\bar{x} + \dots + (x\bar{x})^n$ .
- Let  $\pi : \widehat{X} \longrightarrow X$  be the blow up of  $X$  along a subvariety  $Y$  in  $X$ . Then

$$e(\widehat{X}) = e(X) + e(Y)[x\bar{x} + \dots + (x\bar{x})^r].$$

As an application of the above lemma we will compute the generalized Euler polynomial of  $X$  for a projective hypersurface on  $\mathbb{P}^4$  of degree  $d$  with precisely  $m$  nodes as the singular locus  $\Sigma$ . To fix notation, let  $\widehat{\mathbb{P}^4}$  be the blow up of  $\mathbb{P}^4$  along  $\Sigma$ ,  $\widehat{X}$  be the inverse image of  $X$  on  $\widehat{\mathbb{P}^4}$  and  $\widetilde{X}$  be the strict transform of  $X$ . Further, let  $\widehat{\Sigma}$  be the inverse image of  $\Sigma$  and  $\widetilde{\Sigma} = \widehat{\Sigma} \cap \widetilde{X}$ .

Outside the singular locus the blowup is an isomorphism, therefore one has the following quasi-projective varieties:

$$X - \Sigma \stackrel{\text{def}}{=} W \cong \widehat{W} \stackrel{\text{def}}{=} \widehat{X} - \widehat{\Sigma} \cong \widetilde{X} - \widetilde{\Sigma} \stackrel{\text{def}}{=} \widetilde{W}.$$

Now, we recall Bott's theorem ([7] proposition 10.11):

$$H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} 0 & \text{for } p \neq q, \\ \mathbb{C} & \text{for } p = q \leq n \end{cases}$$

and in particular for  $n = 4$ :  $e(\mathbb{P}^4) = 1 + x\bar{x} + x^2\bar{x}^2 + x^3\bar{x}^3 + x^4\bar{x}^4$ . Also

$$e(\widehat{\mathbb{P}^4}) = e(\mathbb{P}^4) + e(\Sigma)(x\bar{x} + \dots + \dots + (x\bar{x})^3)$$

using that  $e(\Sigma) = m$  and substituting in the above formula:

$$h^{p,q}(\widehat{\mathbb{P}^4}) = \begin{cases} 0 & \text{if } p \neq q, \\ 1 & \text{if } p = q = 0, \\ m + 1 & \text{if } 1 \leq p = q \leq 3, \\ 1 & \text{if } p = q = 4. \end{cases}$$

It follows that  $h^{1,1}(\widehat{\mathbb{P}^4}) = h^{2,2}(\widehat{\mathbb{P}^4}) = h^{3,3}(\widehat{\mathbb{P}^4}) = m + 1$ . After these basic preliminaries, we finally compute:

**Lemma 12.**  $e(X) = 1 + (1 - m)x\bar{x} + ax^3 + (b - m)x^2\bar{x} + (b - m)x\bar{x}^2 + a\bar{x}^3 + x^2\bar{x}^2 + x^3\bar{x}^3$ , where  $a = h^{3,0}(Y)$ ,  $b = h^{2,1}(Y)$  and  $Y$  is a smooth hypersurface of degree  $d$ .

PROOF. Observe that  $\widehat{\Sigma} = \cup_{x \in \Sigma} E_x$  and by cutting each  $E_x$  with  $\widetilde{X}$  we obtain a quadric surface  $Q_x$  hence  $e(\widetilde{\Sigma}) = \sum_x e(Q_x)$  but each summand is equal to  $e(\mathbb{P}^1 \times \mathbb{P}^1) = e(\mathbb{P}^1)^2 = 1 + 2x\bar{x} + x^2\bar{x}^2$ , so  $e(\widetilde{\Sigma}) = m(1 + 2x\bar{x} + x^2\bar{x}^2)$ . Moreover,  $e^{p,q}(\widetilde{W}) = e^{p,q}(\widetilde{X}) - e^{p,q}(\widetilde{\Sigma})$  and  $e^{p,q}(X) = e^{p,q}(W) + e^{p,q}(\Sigma) = e^{p,q}(\widetilde{W}) + e^{p,q}(\Sigma) = e^{p,q}(\widetilde{X}) - e^{p,q}(\widetilde{\Sigma}) + e^{p,q}(\Sigma)$ . Since  $h^3(\widetilde{X}) = h^{3,0}(Y) + h^{2,1}(Y) - m + h^{1,2}(Y) - m + h^{0,3}(Y)$ , where  $Y$  is a smooth hypersurface of degree  $d$  (see proof of lemma 9), taking  $b = h^{1,2}(Y)$  and using the Lefschetz hyperplane theorem:

(4)

$$e(\widetilde{X}) = 1 + (m + 1)x\bar{x} + ax^3 + (b - m)x^2\bar{x} + (b - m)x\bar{x}^2 + a\bar{x}^3 + (1 + m)x^2\bar{x}^2 + x^3\bar{x}^3.$$

Finally,  $e(X) = e(\widetilde{X}) - (m + 2mx\bar{x} + mx^2\bar{x}^2) + m$ . The result follows directly by substituting the value of  $e(\widetilde{X})$  in equation 4. Q.E.D.

## 2. MIXED HODGE STRUCTURE OF A NODAL 3-FOLD

Let  $X, \widetilde{X}, \widehat{X}, W, \widetilde{W}, \widehat{W}, \Sigma, \widetilde{\Sigma}$  and  $\widehat{\Sigma}$  be as in the preceding section, then one has exact sequences of cohomology with compact support:

$$\begin{array}{ccccccc} \dots \longrightarrow & H_c^i(W, \mathbb{C}) & \longrightarrow & H_c^i(X, \mathbb{C}) & \longrightarrow & H_c^i(\Sigma, \mathbb{C}) & \longrightarrow & H_c^{i+1}(W, \mathbb{C}) & \longrightarrow \dots \\ (5) & \cong \downarrow \tilde{\pi}^* & & \downarrow \tilde{\pi}^* & & \downarrow \tilde{\pi}^* & & \cong \downarrow \tilde{\pi}^* & \end{array}$$

$$\dots \longrightarrow H_c^i(\widetilde{W}, \mathbb{C}) \longrightarrow H_c^i(\widetilde{X}, \mathbb{C}) \longrightarrow H_c^i(\widetilde{\Sigma}, \mathbb{C}) \longrightarrow H_c^{i+1}(\widetilde{W}, \mathbb{C}) \longrightarrow \dots$$

where, since  $X, \Sigma, \widetilde{X}$  and  $\widetilde{\Sigma}$  are projective, one can replace the corresponding groups with compact support with the usual cohomology groups.

Since  $\Sigma$  consist of points, then  $H^i(\Sigma, \mathbb{C}) = 0$  for  $i > 0$  and so  $H^i(X, \mathbb{C}) \cong H_c^i(W, \mathbb{C})$  for all  $i \geq 2$ .

Moreover, since all the morphisms involved above are morphisms of MHS, they

are strict and therefore the sequences above remain exact after taking weight filtration, Hodge filtration and the corresponding graded parts, so one has long exact sequences:

$$\dots \longrightarrow \mathrm{Gr}_k^W H_c^i(\widetilde{W}, \mathbb{C}) \longrightarrow \mathrm{Gr}_k^W H^i(\widetilde{X}, \mathbb{C}) \longrightarrow \mathrm{Gr}_k^W H^i(\widetilde{\Sigma}, \mathbb{C}) \longrightarrow \dots$$

for all  $k, i$ .

Now  $\widetilde{X}$  is smooth and  $\widetilde{\Sigma}$  is a disjoint union of smooth quadrics in this case, therefore their Hodge structures are pure and consequently for  $k = 3$  one gets:

$$0 \longrightarrow \mathrm{Gr}_3^W H_c^3(\widetilde{W}, \mathbb{C}) \longrightarrow H^3(\widetilde{X}, \mathbb{C}) \longrightarrow H^3(\widetilde{\Sigma}, \mathbb{C}) \longrightarrow \dots$$

and since  $H^3$  of a quadric is zero and  $H^i(X, \mathbb{C}) \cong H_c^i(\widetilde{W}, \mathbb{C})$  for  $i \geq 2$ , this says that

$$(6) \quad \mathrm{Gr}_3^W H^3(X, \mathbb{C}) \longrightarrow H^3(\widetilde{X}, \mathbb{C})$$

is an isomorphism.

In particular this means that the weight filtration for  $H^3(X, \mathbb{C})$  satisfies:

$$0 = W_1 H^3(X, \mathbb{C}) \subset W_2 H^3(X, \mathbb{C}) \subset W_3 H^3(X, \mathbb{C}) = H^3(X, \mathbb{C})$$

and

$$\mathrm{Gr}_3^W H^3(X, \mathbb{C}) \cong H^3(\widetilde{X}, \mathbb{C}).$$

Since  $H^i(\widetilde{\Sigma}, \mathbb{C}) = 0$  for all  $i = 2k + 1$ , equation (5) gives us an exact sequence:

$$(7) \quad 0 \longrightarrow H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{X}, \mathbb{C}) \longrightarrow H^2(\widetilde{\Sigma}, \mathbb{C}) \longrightarrow H^3(X, \mathbb{C}) \longrightarrow H^3(\widetilde{X}, \mathbb{C}) \longrightarrow 0$$

As the Hodge structures associated to  $\widetilde{X}$  and  $\widetilde{\Sigma}$  are pure, this gives rise to:

$$0 \longrightarrow H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{X}, \mathbb{C}) \longrightarrow H^2(\widetilde{\Sigma}, \mathbb{C}) \longrightarrow W_2 H^3(X, \mathbb{C}) \longrightarrow 0$$

and clearly we have a short exact sequence of MHS:

$$(8) \quad 0 \longrightarrow W_2 H^3(X, \mathbb{C}) \longrightarrow H^3(X, \mathbb{C}) \longrightarrow H^3(\widetilde{X}, \mathbb{C}) \longrightarrow 0$$

As a consequence of the proof of lemma 12 one obtains  $H^2(\widetilde{X}, \mathbb{C}) \cong \mathbb{C}^{m+1}$  and  $H^2(\widetilde{\Sigma}, \mathbb{C}) \cong \mathbb{C}^{2m}$ .



For sake of completeness consider the weight two part of the exact sequence of compact support associated to  $\widetilde{W}$ :

$$(9) \quad 0 \longrightarrow W_2 H_c^2(\widetilde{W}, \mathbb{C}) \longrightarrow W_2 H^2(\widetilde{X}, \mathbb{C}) \longrightarrow W_2 H^2(\widetilde{\Sigma}, \mathbb{C}) \longrightarrow \\ \longrightarrow W_2 H_c^3(\widetilde{W}, \mathbb{C}) \longrightarrow W_2 H^3(\widetilde{X}, \mathbb{C}) \longrightarrow 0$$

Since  $\widetilde{X}$  and  $\widetilde{\Sigma}$  are smooth and projective, the Hodge structure of their  $i$ -th cohomology groups are pure of weight  $i$ , therefore  $W_2 H^3(\widetilde{X}, \mathbb{C}) = 0$ . Also,  $H^1(\widetilde{\Sigma}, \mathbb{C}) = 0$  so  $H_c^2(\widetilde{W}, \mathbb{C})$  is a Hodge substructure of  $H^2(\widetilde{X}, \mathbb{C})$  and therefore also pure of weight 2 (and in fact of type (1,1) since  $H^2(\widetilde{X}, \mathbb{C})$  is of type (1,1), see 4), hence the sequence 9 becomes:

$$(10) \quad 0 \longrightarrow H_c^2(\widetilde{W}, \mathbb{C}) \longrightarrow H^2(\widetilde{X}, \mathbb{C}) \longrightarrow H^2(\widetilde{\Sigma}, \mathbb{C}) \longrightarrow W_2 H_c^3(\widetilde{W}, \mathbb{C}) \longrightarrow 0$$

where the last arrow is surjective; so by defining  $s = \dim(W_2 H_c^3(\widetilde{W}))$  and  $l = \dim(H_c^2(\widetilde{W}))$  we obtain from the dimensions calculated at the end of the last paragraph:  $0 \leq l \leq m + 1$  and  $0 \leq s \leq 2m$ . Similarly, since  $H^2(\widetilde{\Sigma}, \mathbb{C})$  is a pure Hodge structure of weight 2 and type (1,1), then so is  $W_2 H_c^3(\widetilde{W}, \mathbb{C})$ , hence  $h^{1,1}(H_c^3(\widetilde{W})) = s$ . We had already computed  $e(W) = e(\widetilde{X}) - e(\widetilde{\Sigma})$  ( see proof of lemma 12). Since we have already computed that  $H^2(\widetilde{\Sigma}) \simeq \mathbb{C}^{2m}$  we can easily compute  $e(\widetilde{\Sigma})$  ( see also 7) hence  $e^{1,1}(W) = 1 - m = l - s$  hence  $m - 1 \leq l + m - 1 = s \leq 2m$ .

**Example 13.** *It is not difficult to see, using the above description, that for a smooth quintic 3-fold  $X \subset \mathbb{P}^4$  one has  $h^{3,0} = h^{0,3} = 1$  and  $h^{2,1} = h^{1,2} = 101$ . Moreover, if  $X_t$  is a smooth family of quintic 3-folds in  $\mathbb{P}^4$  it is possible to show that the Gauss-Manin connection induces a maximal unipotent map on  $H^3(X_t, \mathbb{C})$ , for any  $t$ , whose nilpotent part  $N$  satisfies  $N(H^{3-p,p}) \subset H^{3-p+1,p-1}$  for  $0 \leq p \leq 3$  with  $N^3 \neq 0$  but  $N^4 = 0$ . In particular one has an splitting of the Hodge structure:*

$$H^3(X_t, \mathbb{C}) = H \oplus_{i=1}^{100} V_i(1),$$

where  $H$  is a weight 3 Hodge structure of type (1,1,1,1) and each  $V_i(1)$  is a weight 3 Hodge structure of type (0,1,1,0), associated to a weight one Hodge structure  $V_i$  of type (1,1) (see also [2] for the quintic family of 3-folds in connection with mirror symmetry or see our example 14 below for the precise equation in question ). Here, as usual,  $V_i(1) = V_i \otimes Q(1)$  and  $Q(1)$  is the Tate-Hodge structure of weight 2.

**Example 14.** *We introduce very briefly a few well known facts about symmetric functions, for this our reference is (see [11], chapter 7): Let  $p_k = \sum_0^5 x_i^k$  be the  $k$ -th power sum symmetric function and  $e_5 = \sum_{i_0 < \dots < i_4} x_{i_0} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$  be the fifth elementary symmetric function. As the classical example quite cited in high energy*

physics computations is in [2], where the authors introduced and studied as an example the pencil of hypersurfaces in  $\mathbb{P}^4$  defined by  $f_t = p_5 + te_5$ . In this notation, one can define the standard hyperplane in  $\mathbb{P}^n$  given by  $H = \{p_1 = 0\} = \mathbb{P}^{n-1}$ . Let us restrict to the case  $n = 5$  and introduce the pencil of quintic hypersurfaces in  $H = \mathbb{P}^4$  defined by  $f_{(\alpha, \beta)} = \alpha p_5 - \frac{5(\alpha+\beta)}{6} p_2 p_3$  and the incidence family  $\mathcal{M} \subset H \times \mathbb{P}^1$ . Clearly, for each  $(\alpha, \beta)$  we have a quintic  $\mathcal{M}_{(\alpha, \beta)} \subset \mathbb{P}^4$ . This family has already been introduced and studied in [13]. In loc.cit ( see Theorem 2 ), he shows that for a general value  $(\alpha, \beta)$ ,  $\mathcal{M}_{(\alpha, \beta)}$  has exactly 100 singular points except for 6 points ( compare with the bound  $m \leq 101$  computed in remark 10). As we have already considered,  $h^3(\tilde{X}, \mathbb{C})$  decreases by two for every node ( see 2), with respect to the  $h^3$  of a non-singular quintic, where  $\tilde{X}$  and  $X$  are as in section 1. In this case,  $h^3(\tilde{X}, \mathbb{C}) = 4$  and a straightforward computation shows that this is a pure Hodge structure of weight 3 and type  $(1, 1, 1, 1)$ . Moreover, the long exact sequence 8 together with the discussion in section 2 shows that:

$$(11) \quad \mathrm{Gr}_3^W H^3(X, \mathbb{C}) \cong H^3(\tilde{X}, \mathbb{C})$$

is a pure Hodge structure of weight 3 and type  $(1, 1, 1, 1)$ , while

$$(12) \quad H^2(\tilde{\Sigma}, \mathbb{C}) \twoheadrightarrow W_2 H^3(X, \mathbb{C}).$$

### 3. EQUISINGULAR FAMILIES

Let

$$\begin{array}{ccc} \bar{\mathcal{X}} = V(uF - vG) & \hookrightarrow & \mathbb{P}^4 \times \mathbb{P}^1 \\ & \searrow \bar{f} & \downarrow \pi_2 \\ & & \mathbb{P}^1 \end{array}$$

be a flat family of hypersurfaces on  $\mathbb{P}^4$ , with  $F$  and  $G$  homogeneous polynomials of degree  $d$  and assume that there is a maximal non empty open subset  $B \subset \mathbb{P}^1$  over which the family

$$\begin{array}{ccc} \mathcal{X} = f^{-1}(B) & \hookrightarrow & \bar{\mathcal{X}} \\ \downarrow f & & \downarrow \bar{f} \\ B & \hookrightarrow & \mathbb{P}^1 \end{array}$$

is real analitically trivial and such that the singular locus  $\Sigma_t$  of every fiber  $X_t$  consists of exactly  $m$  nodes. Then the higher direct image sheaf  $R^3 f_* \mathbb{C}$  is a local system, with fiber  $H^3(X_t, \mathbb{C})$ . It is well known that the Hodge filtration associated to the fibers extend to a Hodge filtration of the sheaf  $\mathcal{H} := R^3 f_* \mathbb{C} \otimes \mathcal{O}_B$ , which then becomes a VMHS.

For a fixed  $t \in B$ , let  $\widehat{\mathbb{P}^4}$  be the blow up of  $\mathbb{P}^4$  along  $\Sigma_t$ ,  $\widehat{X}_t$  the inverse image of  $X_t$  and  $\tilde{X}_t$  be the strict transform of  $X_t$ . Further, let  $\widehat{\Sigma}_t$  be the inverse image of  $\Sigma_t$  (i.e., the disjoint union of the exceptional divisors along the  $m$  nodes) and

$\widetilde{\Sigma}_t = \widehat{\Sigma}_t \cap \widetilde{X}_t$ . Since the multiplicity of every point in  $\Sigma_t$  is 2, then  $\widetilde{X}_t$  is a projective, non singular variety and we have a diagram

$$\begin{array}{ccccc} \widehat{\Sigma}_t & \hookrightarrow & \widehat{X}_t & \hookrightarrow & \widehat{\mathbb{P}}^4 \\ \downarrow & & \downarrow \hat{\pi} & & \downarrow \pi \\ \Sigma_t & \hookrightarrow & X_t & \hookrightarrow & \mathbb{P}^4 \\ \uparrow & & \uparrow \tilde{\pi} & & \uparrow \pi \\ \widetilde{\Sigma}_t & \hookrightarrow & \widetilde{X}_t & \hookrightarrow & \widehat{\mathbb{P}}^4 \end{array}$$

with  $\widetilde{X}_t \subset \widehat{X}_t$  and all horizontal arrows are inclusions of closed algebraic subvarieties.

Outside the singular locus the blowup is an isomorphism, therefore one has the open subvarieties:

$$\mathbb{P}^4 - X_t \stackrel{def}{=} U_t \cong \widehat{U}_t \stackrel{def}{=} \widehat{\mathbb{P}}^4 - \widehat{X}_t \hookrightarrow \widehat{\mathbb{P}}^4 - \widetilde{X}_t \stackrel{def}{=} \widetilde{U}_t.$$

Analogously, we have an inclusion of the above mentioned spaces given in the following commutative diagram:

$$\begin{array}{ccccccc} \widetilde{X}_t & \hookrightarrow & \widehat{X}_t & \hookrightarrow & \widehat{\mathbb{P}}^4 & \longleftarrow & \widetilde{U}_t \longleftarrow \widehat{U}_t \\ & \searrow \tilde{\pi} & \downarrow \hat{\pi} & & \downarrow \pi & & \downarrow \pi| \\ & & X_t & \hookrightarrow & \mathbb{P}^4 & \longleftarrow & U_t \cup \Sigma_t \longleftarrow U_t \end{array}$$

and associated to this diagram there is a commutative diagram of long exact sequences of cohomology with compact support:

$$\begin{array}{ccccccc} \dots \longrightarrow & H_c^i(\widetilde{U}_t, \mathbb{C}) & \longrightarrow & H_c^i(\widehat{\mathbb{P}}^4, \mathbb{C}) & \longrightarrow & H_c^i(\widetilde{X}_t, \mathbb{C}) & \longrightarrow H_c^{i+1}(\widetilde{U}_t, \mathbb{C}) \longrightarrow \dots \\ & \uparrow \tilde{\pi}^* & & \uparrow \pi^* & & \uparrow \tilde{\pi}^* & & \uparrow \tilde{\pi}^* \\ \dots \longrightarrow & H_c^i(U_t, \mathbb{C}) & \longrightarrow & H_c^i(\mathbb{P}^4, \mathbb{C}) & \longrightarrow & H_c^i(X_t, \mathbb{C}) & \longrightarrow H_c^{i+1}(U_t, \mathbb{C}) \longrightarrow \dots \\ & \cong \downarrow \hat{\pi}^* & & \downarrow \pi^* & & \downarrow \hat{\pi}^* & & \cong \downarrow \hat{\pi}^* \\ \dots \longrightarrow & H_c^i(\widehat{U}_t, \mathbb{C}) & \longrightarrow & H_c^i(\widehat{\mathbb{P}}^4, \mathbb{C}) & \longrightarrow & H_c^i(\widehat{X}_t, \mathbb{C}) & \longrightarrow H_c^{i+1}(\widehat{U}_t, \mathbb{C}) \longrightarrow \dots \end{array}$$

Since  $\mathbb{P}^4, \widehat{\mathbb{P}}^4, \widehat{X}_t, \widetilde{X}_t$  and  $X_t$  are projective, we can replace the cohomology with compact support with ordinary cohomology for those varieties. Furthermore, every morphism in the diagram is a morphism of MHS, therefore strict compatible with the weight and the Hodge filtration, hence one has a commutative diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H^3(\widehat{\mathbb{P}}^4, \mathbb{C}) & \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H^3(\widetilde{X}_t, \mathbb{C}) & \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H_c^4(\widetilde{U}_t, \mathbb{C}) & \longrightarrow \cdots \\
& \uparrow \pi^* & & \uparrow \tilde{\pi}^* & & \uparrow \tilde{\pi}^* & \\
\cdots \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H^3(\mathbb{P}^4, \mathbb{C}) & \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H^3(X_t, \mathbb{C}) & \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H_c^4(U_t, \mathbb{C}) & \longrightarrow \cdots \\
& \downarrow \pi^* & & \downarrow \hat{\pi}^* & & \cong \downarrow \hat{\pi}^* & \\
\cdots \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H^3(\widehat{\mathbb{P}}^4, \mathbb{C}) & \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H^3(\widehat{X}_t, \mathbb{C}) & \longrightarrow & \mathrm{Gr}_F^3 \mathrm{Gr}_3^W H_c^4(\widehat{U}_t, \mathbb{C}) & \longrightarrow \cdots
\end{array}$$

As  $\mathbb{P}^4$ ,  $\widehat{\mathbb{P}}^4$  and  $\widetilde{X}_t$  are smooth projective varieties, its cohomology groups  $H^k$  are pure HS of weight  $k$ , so

$$\begin{aligned}
\mathrm{Gr}_3^W H^3(\mathbb{P}^4, \mathbb{C}) &= H^3(\mathbb{P}^4, \mathbb{C}), \\
\mathrm{Gr}_3^W H^3(\widehat{\mathbb{P}}^4, \mathbb{C}) &= H^3(\widehat{\mathbb{P}}^4, \mathbb{C}), \\
\mathrm{Gr}_3^W H^3(\widetilde{X}_t, \mathbb{C}) &= H^3(\widetilde{X}_t, \mathbb{C})
\end{aligned}$$

and

$$\mathrm{Gr}_3^W H^4(\widehat{\mathbb{P}}^4, \mathbb{C}) = \mathrm{Gr}_3^W H^4(\mathbb{P}^4, \mathbb{C}) = \mathrm{Gr}_3^W H^4(\widetilde{X}_t, \mathbb{C}) = 0.$$

Observe that  $\dim H^3(\widetilde{X}_t, \mathbb{C}) = \dim H^3(Z, \mathbb{C}) - 2m$ , where  $Z \subset \mathbb{P}^4$  is a general nonsingular hypersurface of degree  $d$ , as mentioned in equation 2.

**Example 15.** Consider the family  $\mathcal{M}_{(\alpha, \beta)} \subset \mathbb{P}^4$  already considered in example 14, where we assume w.l.o.g that  $\alpha \neq 0$  hence by setting  $t = \frac{5(\alpha + \beta)}{6\alpha}$  the defining equation for this family is given by

$$f_t = p_5 - tp_2p_3$$

on  $\mathbb{P}^5$  intersected with the hyperplane  $H$ , inducing a family  $\mathcal{M}_t \subset \mathbb{P}^4$  of quintic 3-folds with exactly 100 nodes outside the 6 points in loc. cit. Then the cohomology groups  $H^3(X_t, \mathbb{C})$  form a VMHS whose graded parts  $\mathrm{Gr}_3^W H^3(X_t, \mathbb{C})$  constitute a VHS of weight 3 isomorphic to the VHS given by the desingularizations  $\widetilde{X}_t$ , for  $t$  outside the 6 special points. Since the monodromy transformation is no longer maximal unipotent (it can be shown that  $N_t^3 = 0$ ), the associated nilpotent transformation  $N_t$  should satisfy  $N_t \neq 0$  but  $N_t^2 = 0$ , see [4]. We will study this situation in a sequel to this paper.

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